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1984 J. Phys. A: Math. Gen. 17 L635

(http://iopscience.iop.org/0305-4470/17/12/002)

J. Phys. A: Math. Gen. 17 (1984) L635–L638. Printed in Great Britain

LETTER TO THE EDITOR

Quantum particle in a hierarchical potential with tunnelling over arbitrarily large scales

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Received 9 May 1984

Abstract. We introduce a notion of hierarchical potential by iterating a basic elementary geometric construction over an increasing sequence of length scales. We then show that a quantum particle in such a potential exhibits a subdiffusive behaviour characterised by $\langle r^2\rangle \leq C(\ln t)^\beta$ if the initial state is a normalised wavepacket superposition of states of sufficiently low energy.

Diffusion of a quantum particle in complicated potentials has attracted considerable attention in recent years on the part of theoretical and mathematical physicists. The reason is that an understanding of this problem is believed to be important in connection with the more general question of the behaviour of disordered systems, in particular of the metal–insulator transition. This idea goes back to Anderson's famous paper (1958) and in the last ten years has stimulated extended studies of the behaviour of quantum particles in quasi periodic and stochastic potentials. This subject turned out to be a difficult one and in spite of an impressive accumulation of results, the basic physical mechanisms underlying the diffusion of a quantum particle in these potentials are not as transparent as one would wish. There are cases where one may reasonably think that diffusion is essentially determined by the possibility of tunnelling through potential barriers over arbitrarily large length scales. This point of view has been developed systematically by Frölich and Spencer (1983) in their study of the Anderson model where they show that tunnelling over large distances is very unlikely in a probabilistic sense. As a consequence they prove that $\lim_{t\to\infty}(\langle r^2\rangle/t)=0$ where $\langle r^2\rangle$ is the squared distance of the particle averaged over the wavefunction and the potential. They also argue that the stronger result $\langle r^2\rangle \leq$ constant should hold in their situation.

Tunnelling is a very subtle phenomenon already at the deterministic level. In previous work (Jana-Lasinio *et al* 1981a, b) we have shown that tunnelling, even when energetically possible in the sense that the potential has many absolute minima (e.g. a periodic function), can be very unstable under small perturbations localised away from the minima. This result adds further evidence to the fact that tunnelling over large distances is a rather exceptional situation. In view of all this, it is of interest to investigate in more detail the relationship between tunnelling and diffusion by considering simplified models where the calculation can be pushed to the end. In this paper

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0305-4470/84/120635+04\$02.25

we construct a class of potentials which exhibit a geometrical hierarchical structure where tunnelling is energetically possible over all scales but diffusion does not take place at low energies even in cases where there are delocalised states. In these models, even if tunnelling is effective over large distances, the scarcity of low energy levels allows only a subdiffusive behaviour such that $\langle r^2 \rangle \leq c(\ln t)\beta$ if the average is taken with respect to wavepackets superposition of states of sufficiently low energy. For certain special models is it reasonable to expect also a lower bound of the same kind at least in one dimension. When stochasticity is introduced we get the expected result $\langle r^2 \rangle \leq$ constant.

An interesting aspect of the models proposed is that they represent a natural domain of application of the techniques developed by Frölich and Spencer, providing thereby an explicit illustration of their effectiveness.

For simplicity we begin by describing a very special example which besides being hierarchical has also symmetries. Let $d_0 = e^{k_0}$, $k_0 > 0$, and set $d_k = d_0^{a^k}$, $a > 1$. In what follows the numbers $\{d_k\}_0^\infty$ play the role of length scales characteristic of the models. Let Λ_k be the cube centred at the origin of side $6d_k$ with its faces parallel to the coordinate axes. We also denote by Λ_k^α, $k = 1, 2, \ldots, 2d$ the cubes obtained by translating Λ_k along the positive and negative directions of the coordinate axes until it reaches the boundary of the cube Λ_{k+1} (see figure 1). By construction dist $(\Lambda_k^\alpha, \Lambda_k^\beta) > 2d_{k+1}$ if $\alpha \neq \beta$. We also set $\Lambda_k^0 \equiv \Lambda_k$.

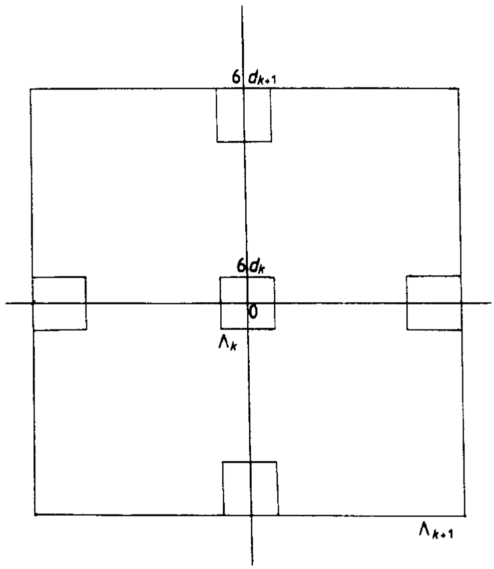

Figure 1.

We now define inductively the potential $V(x)$:

(i) $V(x) = 0$, $x \in \Lambda_0$;

(ii) $V(x) = \lambda > 0$, $x \in \Lambda_{k+1} \setminus \bigcup_{\alpha=0}^{2d} \Lambda_k^\alpha$;

(iii) $V(x)$ in Λ_k^α is the same as in Λ_k^0.

The constant λ has to be sufficiently large so that the spectrum in the box Λ_0 with Dirichlet boundary conditions contains levels below λ.

This model has some special symmetries which are not essential for the main result of this paper. In particular the only important feature is that inside a box Λ_k, the subregions Λ_{k-1}^α where the potential can take the value $V(x)=0$ have diameter of order d_{k-1} and are separated one from the other by a distance of order d_k. In a subsequent paper we shall define a general class of models for which our result applies. The main features of these models are the following. There may be resonances over all scales d_k in the sense that all the boxes Λ_l^α, $l<k$ contained in a region Λ_k, $k=1, 2, \ldots$ may have arbitrarily close eigenvalues below λ when we isolate them one from the other with Dirichlet conditions on their boundary. In the specific model described above they actually have the same spectrum. In these conditions tunnelling is in principle possible over all length scales with the effect of producing delocalised states.

We consider a wavepacket ϕ constructed in the following way. We take a function $\psi(x)$ well localised near the origin and the operator $g(H)$ where g is a positive function with $\operatorname{supp} g \subset [\alpha_1, \alpha_2] \subset [0, \lambda)$ and H is the Hamiltonian. Our wavepacket is $\phi = g(H)\psi$. In words it contains only energies between α_1 and α_2. The quantity

$$\langle r^2 \rangle = \int \mathrm{d}\boldsymbol{x} |(\mathrm{e}^{-\mathrm{i}Ht}\phi)(x)|^2 x^2 \tag{1}$$

is then well defined for any $t \geq 0$. Our main result is that for the models described above

$$\langle r^2 \rangle \leq c(\ln t)^\beta, \qquad \beta > 0, c > 1. \tag{2}$$

We give here an outline of the proof. The details will be given in a subsequent paper of more mathematical character. Using the spectral theorem we can rewrite (1) in the form

$$\langle r^2 \rangle = (2\pi)^{-2} \int \mathrm{d}\boldsymbol{x}\, x^2 \left| \oint_\Gamma \mathrm{e}^{-\mathrm{i}tz}(H-z)^{-1}\phi \,\mathrm{d}z \right|^2. \tag{3}$$

We choose the contour Γ as in figure 2, that is in such a way that the imaginary part shrinks as $1/t$.

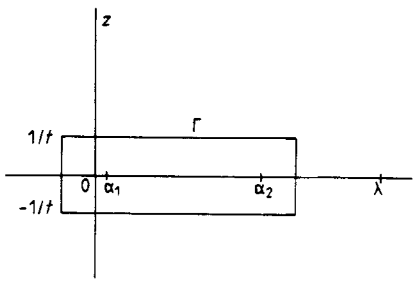

Figure 2.

Consider now a distance $d_{k(t)+1}$ such that $k(t$ is the smallest integer for which

$$\exp(-\sqrt{d_{k(t)}}) \leq 1/t. \tag{4}$$

That is, using the definition of the scales d_k,

$$(\ln t)^{2a} \leq d_{k(t)+1} \leq (\ln t)^{2a^2}. \tag{4'}$$

We now observe that with our choice of Γ and considering the special geometry of the models, the conditions on the exponential decay of the Green function $(H-z)^{-1}$ (x, y) proved by Frölich and Spencer are automatically satisfied on the scale $d_{k(t)+1}$ if we choose as they do $a = \frac{5}{4}$. We remark here that although they prove their result in $\mathbb{Z}^d$, it extends to the continuous case as shown in Holden and Martinelli (1984).

We then split the integral in (3) in two parts: one outside a sphere of radius $d_{k(t)+1}$ which can be bounded by a constant uniformly in t in view of the exponential decay of $(H-Z)^{-1}(x, y)$, and one inside the sphere which can be bounded by $d^2_{k(t)+1}$. We have therefore

$$\langle r^2 \rangle \leqslant c d^2_{k(t)+1} \leqslant c(\ln t)^{4a^2}. \tag{5}$$

In addition to the motivations explained in the introdction we think the models constructed to be of interest in view of the scarcity of examples where such subdiffusive behaviour can be demonstrated. A notable exception is Sinai's result (1982) on the one-dimensional random walk in a random environment where $r \sim \ln^2 t$.

There are a number of additional results that can be proved on the structure of the spectrum below λ for the specific model described earlier. For example it is possible to prove that the Lebesgue measure of the spectrum is zero and that there are no isolated eigenvalues of finite multiplicity. There is also strong evidence for the existence of delocalised states.

Above λ the model exhibits almost free evolution for suitably constructed wavepackets.

If we introduce a stochasticity by letting the bottoms of the wells fluctuate randomly and independently or by keeping the minima of the potential fixed and deforming randomly the shape of the wells, we obtain that all the states of energy below λ are exponentially localised. Furthermore $\langle r^2 \rangle \leqslant$ constant.

As a final remark we would like to emphasise that our conditions of hierarchicity are purely geometrical. Therefore our models could be in principle physically realised. In particular one could conceive of a superlattice (see e.g. Linh 1983) in which the thickness of the layers is arranged in such a way that the hierarchicity conditions are satisfied.

It is a pleasure to thank B Souillard for a very useful discussion. This work was partly done while the authors were visiting the Laboratoire de Physique Théorique et Hautes Energies, Université de Paris VI, the IHES and the University of Trento. We would like to thank all these institutions for thier kind hospitality and financial support.

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